

V. MARKOV'S PROBLEM FOR MONOTONE POLYNOMIALS

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ABSTRACT. We consider the classical problem of estimating norm of the derivative of algebraic polynomial via the norm of polynomial itself. The corresponding extremal problem for general polynomials in uniform norm was solved by V. Markov. In this note we solve analogous problem for monotone polynomials. As a consequence, we find exact constant in Bernstein inequality for monotone polynomials.

1. INTRODUCTION

We consider the following extremal problem:

For a given norm $\|\cdot\|$, determine the best constant A_n such that the inequality

$$\|P'_n\| \leq A_n \|P_n\|$$

holds for all $P_n \in \mathbb{P}_n$, i.e.,

$$A_n = \sup_{P_n \in \mathbb{P}_n} \frac{\|P'_n\|}{\|P_n\|}.$$

The first result in this area appeared in 1889. It is the well-known A. Markov's inequality, namely:

Theorem 1.1. (A. Markov). *For every polynomial $P_n \in \mathbb{P}_n$, the following inequality holds:*

$$(1) \quad \|P'_n\| \leq n^2 \|P_n\|.$$

The equality holds if and only if $P_n = cT_n$, where T_n is the Chebyshev polynomial of the first kind, that is $T_n(x) = \cos(n \arccos x)$ for $x \in [-1, 1]$.

By Δ_n we denote the set of all monotone polynomials of degree n on $[-1, 1]$. In 1926, S. Bernstein [1] pointed out that Markov's inequality for monotone polynomials is not essentially better than for all polynomials, in the sense, that the order of $\sup_{P_n \in \Delta_n} \|P'_n\|/\|P_n\|$ is n^2 . He proved his result only for odd n . In 2001, Qazi [3] extended Bernstein's idea to include polynomials of even degree. Next theorem contains their results:

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Theorem 1.2. (Bernstein [1], Qazi [3]).

$$\sup_{P_n \in \Delta_n} \frac{\|P'_n\|}{\|P_n\|} = \begin{cases} \frac{(n+1)^2}{4}, & \text{if } n = 2k + 1, \\ \frac{n(n+2)}{4}, & \text{if } n = 2k. \end{cases}$$

V. Markov investigated a more general problem:

if k_0, k_1, \dots, k_n are given constants and $P_n(x) = \sum_{i=0}^n a_i x^i$ satisfies $\|P_n\| = 1$, what is the precise bound for the linear form $\sum_{i=0}^n a_i k_i$?

By suitably choosing the constants k_i the linear form can be made equal to any derivative of $P_n(x)$ at any preassigned point.

V. Markov's problem.

Let $x_0 \in [-1, 1]$ be a fixed point. For $0 \leq k \leq n$, find the maximum value of $|P^{(k)}(x_0)|$ over all $P_n \in \mathbb{P}_n$ such that $\|P_n\| = 1$.

The problem was studied more completely and in considerably shorter way by Gusev [2] with the help of a method developed by Voronovskaja, who solved this problem for the case $k = 1$, (see [5]).

In this note, we give a solution of an analogous problem for the case of monotone polynomials and $k = 1$, namely the following problem is considered:

Problem.

Let $x_0 \in [-1, 1]$ be a fixed point. Find the maximum value of $|P'(x_0)|$ over all monotone polynomials $P_n \in \mathbb{P}_n$ such that $\|P_n\| = 1$.

As a consequence, we obtain a simple proof of the main result from [3] as well as sharp Bernstein's inequality for monotone polynomials.

2. PROOF OF MAIN RESULT

In order to formulate the main result the following three types of polynomials are needed :

$$\begin{aligned} S_k(x) &:= (1+x) \sum_{l=0}^k (J_l^{(0,1)}(x))^2; \\ H_k(x) &:= (1-x^2) \sum_{l=0}^{k-1} (J_l^{(1,1)}(x))^2; \\ F_k(x) &:= \sum_{l=0}^k (J_l^{(0,0)}(x))^2. \end{aligned}$$

Theorem 2.1. *Let x_0 be fixed point in the interval $[-1, 1]$. Then, for every $P_n \in \Delta_n$, $n \geq 1$, the following sharp inequality holds:*

$$|P'_n(x_0)| \leq 2 \max(S_k(x_0), S_k(-x_0)) \|P_n\|,$$

for $n = 2k + 2$, $k \geq 0$, and

$$|P'_n(x_0)| \leq 2 \max(F_k(x_0), H_k(x_0)) \|P_n\|,$$

for $n = 2k + 1$, $k \geq 0$.

Proof. We start with the solution of the following problem. Fix $x_0 \in [-1, 1]$, find the maximum value of

$$S(P, x_0) := \frac{P(x_0)}{\int_{-1}^1 P(x) dx},$$

over $P \in \mathbb{P}_n^+$, where \mathbb{P}_n^+ denotes the set of all nonnegative on $[-1, 1]$ polynomials of degree at most n . In what follows, we assume that $x_0 \in (-1, 1)$. All the results can be extended to $x_0 = 1$ and $x_0 = -1$ by continuity.

Note, that this maximum value is attained because of the sequentially compactness of our set \mathbb{P}_n^+ .

Let us denote by $P^*(x)$ an extremal polynomial from $\mathbb{P}_{n,1}^+$ with the largest degree and the maximal number of zeros inside the interval $[-1, 1]$. In other words, if

$$Q(x) = \operatorname{argmax}_{P \in \mathbb{P}_{n,1}^+} S(P, x_0),$$

then $\deg P^* \geq \deg Q$, and the number of zeros of Q inside $[-1, 1] \leq$ than the number of zeros of P^* .

We first prove $\deg(P^*) = n$. Indeed, if $\deg(P^*) \leq n - 1$ consider two polynomials:

$$P_1(x) = (1 - x)P^*(x),$$

$$P_2(x) = (1 + x)P^*(x).$$

None of them can be extremal, hence,

$$\frac{P^*(x_0)}{\int_{-1}^1 P^*(x) dx} > \frac{(1 - x_0)P^*(x_0)}{\int_{-1}^1 (1 - x)P^*(x) dx},$$

and

$$\frac{P^*(x_0)}{\int_{-1}^1 P^*(x) dx} > \frac{(1 + x_0)P^*(x_0)}{\int_{-1}^1 (1 + x)P^*(x) dx}.$$

Multiplying both inequalities by common denominators and adding the results up we get

$$\int_{-1}^1 P^*(x) dx > \int_{-1}^1 P^*(x) dx,$$

that provides a contradiction, and so $\deg(P^*) = n$. The next step is to show, that all zeros of $P^*(x)$ lie in the interval $[-1, 1]$. Suppose that this is not the case and write $P^*(x) = P_1(x)P_2(x)$, where all zeros of P_1 lie in $[-1, 1]$ and $P_2(x) > \delta > 0$, for all $x \in [-1, 1]$, and $\deg(P_2) \geq 1$. Note, that for every fixed polynomial h , $\deg(h) \leq \deg(P_2)$ and sufficiently small t all polynomials of the form $Q(x) = P^*(x) + th(x)P_1(x)$ belong to \mathbb{P}_n^+ . Hence, $t = 0$ should be a point of local minimum of the function

$$g(t) = \frac{P^*(x_0) + th(x_0)P_1(x_0)}{\int_{-1}^1 (P^*(x) + th(x)P_1(x))dx}.$$

This implies that $g'(0) = 0$, where

$$g'(0) = \frac{P_1(x_0)h(x_0) \int_{-1}^1 P^*(x)dx - P^*(x_0) \int_{-1}^1 P_1(x)h(x)dx}{\left(\int_{-1}^1 P^*(x)dx \right)^2},$$

and so

$$\int_{-1}^1 P_1(x)(P_2(x)h(x_0) - P_2(x_0)h(x))dx = 0,$$

for all polynomials h with $\deg(h) \leq \deg(P_2)$. Observe, that this equality implies that if $l(x)$ is such that

$$l(x)(x - x_0) = P_2(x)h(x_0) - P_2(x_0)h(x),$$

then if $h(x)$ runs over all polynomials of degree $\leq \deg(P_2)$, then $l(x)$ runs over all polynomials with $\deg(l) \leq \deg(P_2) - 1$. Therefore,

$$(2) \quad \int_{-1}^1 P_1(x)(x - x_0)l(x)dx = 0$$

holds for all polynomials $l(x)$ of degree $\leq \deg(P_2) - 1$.

If $\deg(P_2) \geq 2$ take $l(x) = x - x_0$ to get a contradiction (integral of a nonnegative non-zero function cannot be equal to 0). Now, suppose that $\deg(P_2) = 1$. Then

$$\int_{-1}^1 P_1(x)(x - x_0)dx = 0$$

and one can write $P^*(x) = (a - x)P_1(x)$ where $a > 1$ or $P^*(x) = (b + x)P_1(x)$ for some $b > 1$. In both of these cases it is easy to see that $S(P^*, x_0) = S(P_1, x_0)$.

Indeed, in the first case

$$\begin{aligned} S(P^*, x_0) &= \frac{(a - x_0)P_1(x_0)}{\int_{-1}^1 (a - x)P_1(x)dx} \\ &= \frac{(a - x_0)P_1(x_0)}{\int_{-1}^1 (x_0 - x)P_1(x)dx + (a - x_0) \int_{-1}^1 P_1(x)dx} = S(P_1, x_0). \end{aligned}$$

In the second case, it can be done in the same way. But then, taking

$$P_3(x) = (1 + x)P_1(x)$$

and

$$P_4(x) = (1 - x)P_1(x)$$

and repeating all arguments from the beginning of the proof one get that either $S(P_3, x_0)$ or $S(P_4, x_0)$ is not less then $S(P, x_0) = S(P_1, x_0)$ and all zeros of P_3 and P_4 lie in the segment $[-1, 1]$, that contradicts our assumption. Hence, all zeros of $P^*(x)$ lie in the interval $[-1, 1]$.

We distinguish two cases depending on parity of n .

If $n = 2k + 1$, $k \geq 0$ an extremal polynomial can be expressed in one of the following ways: $P^*(x) = (1 + x)g^2(x)$ or $P^*(x) = (1 - x)g_1^2(x)$. If $n = 2k$, then an extremal polynomial can be expressed as $P^*(x) = (1 - x^2)g^2(x)$ or $P^*(x) = g^2(x)$. In general, we can write an extremal polynomial as $P^*(x) = w(x)g^2(x)$, where $w(x)$ is one of the function $1 - x, 1 + x, 1 - x^2, 1$.

For any fixed polynomial $h(x)$ with $\deg(h) \leq \deg(g)$ consider the function

$$\psi(t) = \frac{w(x_0)(g(x_0) + th(x_0))^2}{\int_{-1}^1 w(x)(g(x) + th(x))^2 dx}.$$

Since P^* is extremal, this function has a local maximum at $t = 0$, and so $\psi'(0) = 0$, i.e.,

$$(3) \quad \psi'(0) = 2w(x_0) \cdot \frac{g(x_0)h(x_0) \int_{-1}^1 w(x)g^2(x)dx - g^2(x_0) \int_{-1}^1 w(x)g(x)h(x)dx}{\left(\int_{-1}^1 w(x)g^2(x)dx \right)^2} = 0.$$

Since $g(x_0) \neq 0$ (otherwise, $\psi(0) = 0$, which contradicts to maximality of P^*) last equality implies

$$h(x_0) \int_{-1}^1 w(x)g^2(x)dx - g(x_0) \int_{-1}^1 w(x)g(x)h(x)dx = 0$$

or

$$(4) \quad \int_{-1}^1 w(x)g(x)(h(x_0)g(x) - h(x)g(x_0))dx = 0$$

for all polynomials $h(x) \in \mathbb{P}_k$ if $w(x) = 1, 1 - x, 1 + x$ and for all polynomials $h(x) \in \mathbb{P}_{k-1}$, if $w(x) = 1 - x^2$. We first consider the case $w(x) = 1, 1 - x, 1 + x$. Repeating the same argument as we used to prove (2) we can deduce that (4) implies that for all $l \in \mathbb{P}_{k-1}$ we have

$$\int_{-1}^1 w(x)g(x)(x - x_0)l(x)dx = 0.$$

Denote

$$G(x) = (x - x_0)g(x)$$

and consider the sequence of polynomials p_k orthonormal on $[-1, 1]$ with respect to the weight $w(x)$. Since $\deg(G) = k + 1$ and orthonormal polynomials of degree $\leq k + 1$ form a basis (over \mathbb{R}) of \mathbb{P}_{k+1} , one can write

$$G(x) = \sum_{m=0}^{k+1} c_m p_m(x)$$

for some real constants c_m . Taking $l(x) = p_i(x)$ for $0 \leq i \leq k - 1$ we obtain that $c_i = 0$ for $0 \leq i \leq k - 1$. Indeed, if $l(x) = p_i(x)$, $0 \leq i \leq k - 1$ then

$$\int_{-1}^1 w(x)G(x)J_i(x)dx = 0 = \sum_{m=0}^{k+1} c_m \int_{-1}^1 w(x)p_k(x)p_i(x)dx = c_i.$$

Thus,

$$G(x) = (x - x_0)g(x) = c_{k+1}p_{k+1}(x) + c_k p_k(x).$$

Letting $x = x_0$, we get $c_{k+1}p_{k+1}(x_0) + c_k p_k(x_0) = 0$, and so

$$g(x) = g_{extr}(x) := c \frac{p_{k+1}(x)p_k(x_0) - p_{k+1}(x_0)p_k(x)}{x - x_0},$$

for some real constant c . In case $w(x) = 1 - x^2$, we have to take $k - 1$ instead k . It gives us a polynomial g , of the form

$$g(x) = g_{extr}(x) := c \frac{p_k(x)p_{k-1}(x_0) - p_k(x_0)p_{k-1}(x)}{x - x_0}, \quad k \geq 1.$$

Now, using Christoffel-Darboux's formula (see [4]) $S(P, x_0)$ can be computed explicitly. Indeed,

$$\begin{aligned} \int_{-1}^1 w(x) \left(\frac{p_{k+1}(x)p_k(x_0) - p_{k+1}(x_0)p_k(x)}{x - x_0} \right)^2 dx &= \frac{\gamma_k^2}{\gamma_{k+1}^2} \int_{-1}^1 w(x) \sum_{i=0}^k (p_i(x_0)p_i(x))^2 dx \\ &= \frac{\gamma_k^2}{\gamma_{k+1}^2} \sum_{i=0}^k p_i(x_0)^2, \end{aligned}$$

hence

$$\begin{aligned} S(P, x_0) &= w(x_0) \frac{(g_{extr}(x_0))^2}{\int_{-1}^1 w(x)(g_{extr}(x))^2 dx} \\ &= w(x_0) \frac{(\frac{\gamma_{k+1}}{\gamma_k})^2 (\sum_{l=0}^k p_l^2(x_0))^2}{\int_{-1}^1 w(x) \left(\frac{p_{k+1}(x)p_k(x_0) - p_{k+1}(x_0)p_k(x)}{x - x_0} \right)^2 dx} \\ &= w(x_0) \left(\sum_{l=0}^k p_l^2(x_0) \right). \end{aligned}$$

In case, when $n = 2k$ and $w(x) = 1 - x^2$ we get

$$S(P, x_0) = w(x_0) \left(\sum_{l=0}^{k-1} p_l^2(x_0) \right).$$

Let $n = 2k + 1$. If $w(x) = 1 + x$, then

$$p_k(x) = J_k^{(1,0)}(x),$$

where $J_k^{(1,0)}(x)$ is the Jacobi polynomial associated with weight $(1+x)(1-x)$. Hence,

$$S(P, x_0) = (1 + x_0) \sum_{l=0}^k (J_l^{(0,1)}(x_0))^2 = S_k(x_0).$$

By analogy, if $w(x) = 1 - x$, then

$$S(P, x_0) = (1 - x_0) \sum_{l=0}^k (J_l^{(1,0)}(x_0))^2 = S_k(-x_0).$$

In this way, we get sharp pointwise inequality

$$(5) \quad P_{2k+1}(x_0) \leq \max(S_k(x_0), S_k(-x_0)) \int_{-1}^1 P_{2k+1}(x) dx$$

for all $P_{2k+1} \in \mathbb{P}_{2k+1}^+$.

In case $n = 2k$, $w(x) = 1$ or $w(x) = 1 - x^2$ and

$$S(P, x_0) = F_k(x_0)$$

or

$$S(P, x_0) = H_k(x_0)$$

respectively, where H_k and F_k . We arrive at the following sharp pointwise inequality:

$$(6) \quad P_{2k}(x_0) \leq \max(F_k(x_0), H_k(x_0)) \int_{-1}^1 P_{2k}(x) dx$$

for all $P_{2k} \in \mathbb{P}_{2k}^+$.

Let P_n be a polynomial of degree n , that is monotone on $[-1, 1]$, i.e., $P_n \in \Delta_n$. Then, P'_n is a nonnegative polynomial on $[-1, 1]$. Note, that

$$\int_{-1}^1 P'_n(x) dx = P_n(1) - P_n(-1) \leq 2\|P_n\|.$$

Combining last inequality with (5) and (6) we get

$$(7) \quad P'_{2k+2}(x) \leq 2 \cdot \max(S_k(x), S_k(-x)) \|P_{2k+2}\|,$$

and

$$(8) \quad P'_{2k+1}(x) \leq 2 \cdot \max(H_k(x), F_k(x)) \|P_{2k+1}\|,$$

for all monotone polynomials P_n . This completes the proof. \square

Using Theorem 2.1 one can give an alternative proof of Bernstein's result, that is Theorem 1.2 for polynomials of even degree. The following fact about orthogonal polynomials is needed.

Theorem 2.2. (*Szegő [4], 1919*). *Let $w(x)$ be a weight function which is non-decreasing (non-increasing) in the interval $[a, b]$, b and a are finite. If $\{p_n\}$ is the set of the corresponding orthogonal polynomials, the functions $w(x)p_n(x)^2$ attain their maxima in $[a, b]$ at $x = b$ ($x = a$).*

Proof of Theorem 1.2. We consider an even case $n = 2k + 2$, $k \geq 0$. Using Szegő's theorem for non-decreasing weight $w(x) = 1 + x$ and for non-increasing weight $w(x) = 1 - x$ together with the fact that

$$J_l^{(0,1)}(1) = J_l^{(1,0)}(-1) = \frac{\sqrt{l+1}}{\sqrt{2}}$$

we get:

$$\begin{aligned} (1+x)(J_l^{(0,1)}(x))^2 &\leq 2(J_l^{(0,1)}(1))^2 = l+1, \\ (1+x)(J_l^{(1,0)}(x))^2 &\leq 2(J_l^{(1,0)}(-1))^2 = l+1 \end{aligned}$$

for all $l \geq 0$ and $x \in [-1, 1]$. Summing these inequalities for $0 \leq l \leq k$ and using Theorem 2.1 we get

$$P'_n(x) \leq 2 \sum_{l=0}^k (1+l) \|P_n\| = \frac{n(n+2)}{4} \|P_n\|,$$

that proves Bernstein-Markov's inequality for monotone polynomials of even degree.

Multiplying both sides of (7) and (8) by $\sqrt{1-x^2}$ and taking supremum over all $x \in [-1, 1]$ we get the following

Corollary 2.3. (*Sharp Benstein-type inequality for monotone polynomials*).

$$(9) \quad \sup_{P_n \in \Delta_n} \frac{\|P'_n(x)\sqrt{1-x^2}\|}{\|P_n\|} = \begin{cases} 2\|\sqrt{1-x^2}S_k(x)\|, & n = 2k+2, \\ 2\max(\|\sqrt{1-x^2}H_k(x)\|, \|\sqrt{1-x^2}F_k(x)\|), & n = 2k+1. \end{cases}$$

Using estimates for Jacobi polynomials one can observe that the right hand side is of the order $\frac{2}{\pi}n$. This implies that Bernstein's inequality for monotone polynomials is not essentially better than the classical one.

Remark 2.4. From the proof of Theorem(2.1) it follows that equality in (9) holds for one of the following polynomials

$$\begin{aligned} s_k(x) &:= \int_{-1}^x (1+t) \sum_{l=0}^k (J_l^{(0,1)}(t))^2; \\ h_k(x) &:= \int_{-1}^x (1-t^2) \sum_{l=0}^{k-1} (J_l^{(1,1)}(t))^2; \\ f_k(x) &:= \int_{-1}^x \sum_{l=0}^k (J_l^{(0,0)}(t))^2, \end{aligned}$$

that are normalized, such that

$$\begin{aligned} s_k(-1) &= -s_k(1), \\ h_k(-1) &= -h_k(1), \\ f_k(-1) &= -f_k(1). \end{aligned}$$

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